Algebra 2 Segment 1 – Lesson Summary Notes

For each lesson:

- ✓ Read through the LESSON SUMMARY which is located.
- ✓ Read and work through every page in the LESSON.
- ✓ Try each PRACTICE problem and write down the examples on the back of your lesson summary with the worked out solutions or on loose leaf paper to put in your MATH BINDER.
- ✓ Read the directions carefully for the assignment and submit the lesson ASSESSMENT.
- ✓ If you need to RESUBMIT (you see a BLUE ARROW) an assignment, make sure to view the feedback on the assignment by clicking on the assignment in your GRADES before resubmitting.

For more resources go to: http://eschoolmath.weebly.com/algebra-2.html

1.01 Lesson Summary

1. To simplify numerical expressions, use the order of operations: **P**arentheses, **E**xponents, **M**ultiplication/**D**ivision, **A**ddition/**S**ubtraction

   To help remember the order of operations, think: PEMDAS (**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally).

Two expressions are simplified using the order of operations.

$-4 + 12 \div 2 - (9 - 15)^2$	$(7 - 12) + 2^3 \bullet 6 - 10$
$-4 + 12 \div 2 - (-6)^2$	$-5 + 2^3 \bullet 6 - 10$
$-4 + 12 \div 2 - 36$	$-5 + 8 \bullet 6 - 10$
$-4 + 6 - 36$	$-5 + 48 - 10$
$2 - 36$	$43 - 10$
-34	33

2. To solve a one-step equation, remember to do the opposite to isolate the variable.

Six equations are solved using inverse operations.

$x + \cancel{14} = 9$ $-\cancel{14} \quad -14$ $x = -5$	$x - \cancel{8} = -1$ $+\cancel{8} \quad +8$ $x = 7$	$8x = 14$ $\frac{\cancel{8}x}{\cancel{8}} = \frac{14}{8}$ $x = 1.75$ or $1\frac{3}{4}$
$\frac{x}{6} = 1.5$ $\cancel{6} \cdot \left(\frac{x}{\cancel{6}}\right) = 6(1.5)$ $x = 9$	$\sqrt{x} = 25$ $\left(\sqrt{x}\right)^2 = 25^2$ $x = 625$	$x^2 = 36$ $\sqrt{x^2} = \sqrt{36}$ $x = \pm 6$

3. When solving two-step equations, remember to do the following:
   - Read the equation and decide what operations (add, subtract, multiply, divide) are being applied to the variable.
   - Use the reverse of order of operations to undo each of these operations one at a time.
   - You are finished when the variable is isolated on one side of the equation by itself.

$$
\begin{aligned}
4x + \cancel{1} &= 21 \\
-\cancel{1} &\quad -1 \\
4x &= 20 \\
\frac{\cancel{4}x}{\cancel{4}} &= \frac{20}{4} \\
x &= 5
\end{aligned}
$$

4. When the variable occurs on both sides of the equation, isolate the variable terms on one side of the equation and the numbers on the other side.

$$
\begin{aligned}
3x - 11 &= 7 + 9x \\
-3x \quad &= \quad -3x \\
-11 &= \cancel{7} + 6x \\
-7 &\quad -\cancel{7} \\
\frac{-18}{6} &= \frac{\cancel{6}x}{\cancel{6}} \\
-3 &= x \\
&\text{or} \\
x &= -3
\end{aligned}
$$

1.02 Lesson Summary

A function is an equation where each x-coordinate has, at most, one corresponding y-coordinate. The x-coordinates, or inputs, of the function make up the domain while the y-coordinates, or outputs, of the function make up the range.

Graphically, you can determine if an equation is a function by performing the Vertical Line Test. Move a vertical line across the coordinate plane. If the line intersects the graph in more than one place, the equation is not a function.

An equation that is also a function is indicated by the functional notation f(x), which is read “f of x.” To evaluate a function at a given x value, substitute the value into every instance of x in the function.

Operations on Functions

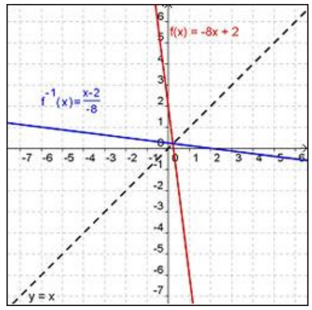
Operations on functions may also be performed. To add two functions, the notation f(x) + g(x) is used. To subtract two functions, use the functional notation f(x) – g(x). Multiplication of two functions is indicated by a closed dot between the function notations: f(x) • g(x).

Inverses of Functions

Follow the steps outlined below to algebraically find the inverse of a function. The notation used to indicate the inverse of a function is $f^{-1}(x)$.

1. Substitute f(x) with y.
2. Reverse the x and y variables.
3. Solve for y.
4. Replace y with $f^{-1}(x)$.

Graphically, when a function is reflected across the line y = x, the new function is said to be the inverse of the function.



Composition of Functions

The composition of functions involves substituting one function in place of the variable for another function. The notation for the composition of two functions is $f(g(x))$ or $(f \circ g)(x)$.

Let $f(x) = x^2 - 4x - 7$ and $g(x) = x - 3$. Find $f(g(x))$.

Work from the inside out by substituting the expression for the function $g(x)$ into the function for $f(x)$.

$f(g(x))=f(x - 3)$

$f(x - 3)=(x - 3)^2 - 4(x - 3) - 7$

Simplify the expression

$f(x - 3)=x^2 - 6x + 9 - 4(x - 3) - 7$

$f(x - 3)=x^2 - 6x + 9 - 4x + 12 - 7$

$f(x - 3)=x^2 - 10x + 14$

$f(g(x))=x^2 - 10x + 14$

To evaluate the function for a given x value, substitute the value for x into the composite function.

Let $f(x) = x^2 - 4x - 7$ and $g(x) = x - 3$. Find $f(g(2))$.

$f(g(x))=x^2 - 10x + 14$

$f(g(2))=(2)^2 - 10(2) + 14$

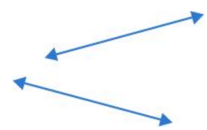
$f(g(2))=4 - 10(2) + 14$

$f(g(2))=4 - 20 + 14$

$f(g(2))=-2$

01.04 Lesson Summary

There are three types of lines: vertical, horizontal, and slanted.

Equation	Type of Line	Graph
$x = a$ number	**Vertical** Think about the "vertebrae" in your back. You stand up and down and so do "vertical" lines.	
$y = a$ number	**Horizontal** Think about the sun rising on the "horizon." A horizon goes left and right and so do horizontal lines.	
$ax + by = c$	**Slanted Line – Rising or Falling** These lines are neither vertical nor horizontal.	

To graph a linear equation using a table of values pick any three values to substitute into the equation for x or y. It's your choice! Then solve for the other variable. Find three ordered pairs. Plot and connect these three points to graph the line.

Given two ordered pairs on a line (x_1, y_1) and (x_2, y_2), the slope of the line (m) is given by the formula:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{difference in y-coordinates}}{\text{difference in x-coordinates}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Slope can also be referred to as the average rate of change. Examples of average rate of change are the speed of a car, salary per week, or steepness of a hill.

line direction	slope
"rising"	positive
"falling"	negative
horizontal	zero
vertical	undefined

The slope-intercept form of a line is $y = mx + b$, where m is the slope and b is the y-intercept. Always start by graphing the y-intercept first. Then count up and to the right if the slope is positive or down and to the right if the slope is negative to plot the second point.

If the equation of a line is changed into an inequality, the graphed line is either solid (for $\geq$ and $\leq$) or dashed (for $>$ and $<$). The graph will be shaded either above or below the line. To determine which, test the point (0, 0) in the inequality.

01.05 Lesson Summary

There are three forms used to write the equation of a line.

Form	Equation
Standard	$ax + by = c$, where a, b, and c are integers and a is positive
Slope-intercept	$y = mx + b$, where m is the slope and b is the y-intercept
Point-slope	$y - y_1 = m(x - x_1)$, where m is the slope and (x_1, y_1) is any point on the line

The standard form of a linear equation cannot be calculated directly from a pair of points, the slope, or y-intercept. This data must be written in one of the other forms first and then transformed into standard form.
Point-slope form is the easiest equation to use when given two points. First calculate the slope, then substitute the slope value and either of the two given points into the equation. Simplify your answer to the requested form.

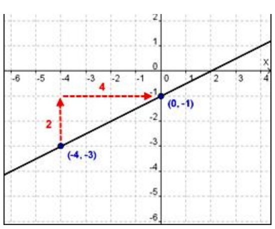
(1, 2) and (−2, 5) $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 2}{-2 - 1} = \frac{3}{-3} = -1$

$$y - y_1 = m(x - x_1)$$
$$y - 2 = -1(x - 1) \text{ (point-slope form)}$$

$$y - 2 = -x + 1$$
$$y = -x + 1 + 2$$
$$y = -x + 3 \text{ (slope-intercept form)}$$

$$y - 2 = -x + 1$$
$$x + y = 1 + 2$$
$$x + y = 3 \text{ (standard form)}$$

To write the equation of a line from the graph, look for two intersection points on the grid. Write down the coordinates of one point, and then use the graph to count the slope value.



Point-Slope Form: $y + 3 = \frac{1}{2}(x + 4)$

Slope-Intercept Form: $y = \frac{1}{2}x - 1$

Standard Form: $x - 2y = 2$

1.06 Lesson Summary

When evaluating linear functions, the key features that need to be identified are:

- The slope—the rate of change of the function
- The y-intercept—what the function equals when the x value is zero
- Increasing or decreasing—from left to right, does the function go up or down?

Ways to represent a function

- Algebraically (for example: f(x) = 2x + 3)
- Graphically
- Table
- Verbally (like a story)

2.01 Lesson Summary

Rational Exponents → Radical Expressions

The numerator of the rational exponent becomes the exponent on the radicand.
The denominator of the rational exponent becomes the index, or root, of the radical.

$$t^{\frac{3}{4}} = \sqrt[4]{t^3}$$

Radical Expressions → Rational Exponents

The exponent on the radicand becomes the numerator of the rational exponent. The index, or root, of the radical becomes the denominator of the rational exponent.

$$\sqrt[8]{w^5} = w^{\frac{5}{8}}$$

2.02 Lesson Summary

Product Property

When multiplying two of the same variable, the exponents are added. Fractions must have common denominators before being added or subtracted.

$$d^{\frac{2}{5}} \cdot d^{\frac{1}{2}} = d^{\frac{4}{10}} \cdot d^{\frac{5}{10}} = d^{\frac{9}{10}}$$

Quotient Property

When dividing two of the same variable, the exponents are subtracted. Fractions must have been common denominators before being added or subtracted.

$$\frac{b^{\frac{5}{6}}}{b^{\frac{1}{4}}} = \frac{b^{\frac{10}{12}}}{b^{\frac{3}{12}}} = b^{\frac{10}{12} - \frac{3}{12}} = b^{\frac{7}{12}}$$

2.03 Lesson Summary

Steps to Solving a Radical Equation

$\sqrt{6x+9}+2=11$

1. Isolate the Radical.
Add, subtract, multiply, or divide any terms or factors not included in the radicand.

$\sqrt{6x+9}+2=11$

$\underline{\qquad -2\ \ -2}$

$\sqrt{6x+9}=9$

2. Square Both Sides.
Each side of the radical must be squared to eliminate the radical sign(s).

$\sqrt{6x+9}=9$

$\left(\sqrt{6x+9}\right)^2=(9)^2$

$6x+9=81$

3. Solve for the Variable.

$6x+9=81$

$\underline{\ \ -9\ \ -9}$

$6x=72$

$\frac{6x}{6}=\frac{72}{6}$

$x=12$

4. Check Your Work!
It is extremely important to check your work when solving radical equations. Some answers that appear to be solutions may in fact be extraneous solutions that do not fit the original equation.

2.05 Lesson Summary

Complex Numbers

A complex number consists of two parts: a real number term and an imaginary number term. An imaginary number is defined as

$$i=\sqrt{-1}$$

All negative radicands may be simplified by factoring the imaginary number of $\sqrt{-1}$ first and then simplifying the remaining radical.

$$\sqrt{-45}=\sqrt{-1}\bullet\sqrt{45}=i\sqrt{45}=i\sqrt{9}\bullet\sqrt{5}$$

$$3i\sqrt{5}$$

A repeating pattern is found when the exponents of i are increased by 1. As long as the first four powers of i are known, any power of i may be simplified by dividing the exponent by 4. The remainder of the division problem will tell you which exponent to use.

$i^1=i$ $\qquad$ $i^2=-1$ $\qquad$ $i^3=-i$ $\qquad$ i^4 or $i^0=1$ (no remainder)

)

$i^{27}=-i$ because

$$\begin{array}{r} 6 \\ 4\overline{)\ 27} \\ \underline{-24} \\ 3 \end{array}$$

and $i^3=-i$.

The conjugate of any complex number is found by changing the sign separating the two terms to its opposite sign.

Example: The conjugate of $7+10i$ → $7-10i$

2.06 Lesson Summary

Adding and Subtracting Complex Numbers

To combine complex numbers, you must distribute any coefficients in front of the parentheses. Then identify and combine like terms by combining the real number parts and the imaginary number parts separately.

$$(5 + 4i) + (6 - 2i)$$
$$1(5 + 4i) + 1(6 - 2i)$$
$$5 + 4i + 6 - 2i$$
$$5 + 4i + 6 - 2i$$
$$11 + 2i$$

If the complex number is written as negative radicands, you must factor out the imaginary number before combining like terms.

$$2 + \sqrt{-121} + \sqrt{-9}$$

$$2 + \sqrt{-1} \bullet \sqrt{121} + \sqrt{-1} \bullet \sqrt{9}$$

$$2 + i\sqrt{121} + i\sqrt{9}$$

$$2 + 11i + 3i$$
$$2 + 14i$$

Remember that the Associative Property and the Commutative Property allow the real terms and the imaginary terms to be added to like terms.

Associative Property: Grouping symbols do not affect the outcome. Numbers can be associated by using parentheses or brackets in an addition or subtraction problem and it does not change the solution.
Commutative Property: The order in which an operation is performed does not affect the outcome. Numbers can move around or commute in an addition or subtraction problem and it does not change the solution.

Multiplying Complex Numbers

When two complex numbers are multiplied, each term in the first set of parentheses must be distributed to each term in the second set of parentheses. Be sure to substitute any instances of i^2 with -1.

$$(6 - 3i)(5 + 9i)$$
$$30 + 54i - 15i - 27i^2$$
$$30 + 39i - 27i^2$$
$$30 + 39i - 27(-1)$$
$$30 + 39i + 27$$
$$57 + 39i$$

Multiplying complex numbers follows the **Distributive Property.** When expressions are multiplied, each term in the first expression must be multiplied by each term in the second expression.

Dividing Complex Numbers

The goal when dividing complex numbers is to eliminate any imaginary numbers in the denominator. This is called rationalizing the denominator.

When there is only an imaginary number in the denominator and no real number, this is done by multiplying both the numerator and denominator by *i*. Substitute any instances of i^2 with -1.
If all parts of the complex number are present, both the numerator and the denominator must be multiplied by the conjugate of the denominator.

$$\frac{7}{3-4i}$$

$$\frac{7}{3-4i}\cdot\frac{(3+4i)}{(3+4i)}$$

$$\frac{21+28i}{9+12i-12i-16i^2}$$

$$\frac{21+28i}{9+12i-12i-16(-1)}$$

$$\frac{21+28i}{9+0i+16}=\frac{21+28i}{25}$$

2.08 Lesson Summary

Adding Polynomials

- Distribute any coefficients.
- Combine like terms.

$$(4x^3 + 5x^2 - 2x - 7) + (2x^3 - 6x^2 - 2)$$
$$4x^3 + 5x^2 - 2x - 7 + 2x^3 - 6x^2 - 2$$
$$6x^3 - x^2 - 2x - 9$$

Subtracting Polynomials

- Distribute any coefficients – don't forget to distribute the understood negative one!
- Combine like terms.

$$(9x^2 - 7) - (8x^2 + 2x + 10)$$
$$9x^2 - 7 - 8x^2 - 2x - 10$$
$$x^2 - 2x - 17$$

Multiplying Polynomials

Type of Factors	Description ***Always combine like terms!***	Example

Monomial and Polynomial	Distribute the monomial to each term in the polynomial	$-8(4x^3 + 5x^2 - x + 3)$ $-32x^3 - 40x^2 + 8x - 24$
Two Monomials and Polynomial	Multiply the monomials together then distribute the product to the polynomial.	$(2x)(3)(7x^2 - 4)$ $(6x)(7x^2 - 4)$ $42x^3 - 24x$
Two Binomials	Distribute each term in the first binomial to each term in the second binomial. Also known as FOIL.	$(3x + 7)(x - 2)$ $3x^2 - 6x + 7x - 14$ $3x^2 + x - 14$
Two Polynomials	Take each term in the first polynomial and distribute it to each term in the second polynomial.	$(2x^2 + 7x - 3)(x + 4)$ $2x^3 + 8x^2 + 7x^2 + 28x - 3x - 12$ $2x^3 + 15x^2 + 25x - 12$

Operations on Functions

Operations may also be performed on functions in function notation.

- To add two functions, the notation $f(x) + g(x)$ is used.
- To subtract two functions, use the functional notation $f(x) - g(x)$.
- Multiplication of two functions is indicated by a closed dot between the function notations: $f(x) \bullet g(x)$.

03.01 Lesson Summary

Greatest Common Factor

A greatest common factor of two or more terms is the largest factor that all terms have in common. The greatest common factor of a polynomial should be factored out first before any further factoring is completed.

Example: $3r^6+27r^4+15r^2=3r^2(r^4+9r^2+5)$

When multiplying variables, add the exponents.

$$r^2 \bullet r^4=rr \bullet rrrr=r^6$$

When factoring a GCF, subtract the exponents.

$$\text{To factor } r^2 \text{ from } r^6\text{: } r^{6-2}=r^4$$

$$rrrrrr=(rr)(rrrr)=r^2(r^4)$$

Difference of Squares Binomials

A difference of squares binomial includes a perfect square term subtracted by another perfect square term.

Pattern:

$$a^2 - b^2 = (a+b)(a-b)$$

Example:

$$r^2 - 4 = (r+2)(r-2)$$

Perfect Square Trinomials

A perfect square trinomial is a polynomial of three terms where the first and last terms are perfect squares and the middle term is twice the product of the square roots of those terms.

Pattern:

$$a^2 + 2ab + b^2 = (a+b)^2 \quad \text{OR} \quad a^2 - 2ab + b^2 = (a-b)^2$$

Example:

$$r^2 + 12r + 36 = (r+6)^2 \quad r^2 - 12r + 36 = (r-6)^2$$

Check your factors to see if they can be factored further
Sometimes after an initial factoring, the remaining terms can be factored further.

Example:
$t^8 - 81$
This is a difference of squares.
$(t^4 + 9)(t^4 - 9)$
The factor (t4−9) is also a difference of squares.
$(t^4 + 9)(t^2 + 3)(t^2 - 3)$
Even though there are even powers here, these are not special products so this is factored completely.
Example:
$3w^2 + 15w + 18$
This has a GCF of 3.
$3(w^2 + 5w + 6)$
The trinomial can be factored by looking for factors of 6 that add up to 5.
$3(w+3)(w+2)$
This is completely factored.

03.02 Lesson Summary

Discriminant Information

This is the standard form of a trinomial: **a**x^2 + **b**x + **c**
The Discriminant is: **b²** – 4**ac**
The nature of the roots can be determined by substituting the numbers from the trinomial into the Discriminant and solving.

If the Discriminant is:	Then there will be:
Zero	1 rational solution

Positive Perfect Square	2 rational solutions
Positive Integer, Not a Perfect Square	2 irrational solutions
Negative	2 complex solutions

Steps for Factoring Polynomials with Four Terms
1. Factor the GCF. Determine if there is a GCF among all terms in the original polynomial and factor it, if one exists.
2. Group the terms. Group the polynomial into two pairs—the first two terms and the last two terms.
3. Factor the GCF from each group. Factor the GCF from the first pair of terms and then from the second pair of terms.
4. Factor the common binomial. • If common binomials do not exist, GCFs were factored accurately, and the expression cannot be rearranged, you have a prime polynomial. • Don't forget to check the factors for accuracy by distributing.

Steps for Factoring Trinomials by Grouping
1. Factor the GCF. Determine if there is a GCF among all terms in the original polynomial and factor it, if one exists.
2. Split the middle term. • Multiply the leading coefficient and the constant. • Find factors of this product that sum to the middle coefficient. • Rewrite the polynomial with those factors replacing the middle term.
3. Factor by grouping • The four-term polynomial is split into two groups—the first pair and last pair of terms. Factor the GCF from each pair. • Factor the common binomial. • If common binomials do not exist and GCFs were factored accurately, you may have a prime polynomial.

03.03 Lesson Summary

Sum of Cubes Pattern: $a^3 + b^3 = (a + b)(a^2 – ab + b^2)$
Difference of Cubes Pattern: $a^3 – b^3 = (a – b)(a^2 + ab + b^2)$

Steps to Factoring the Sum/Difference of Cubes

1. **Factor the GCF.**
2. **Identify the cube root of each term.**
   - The cube root of the first term will represent a in the pattern.
   - The cube root of the second term will represent b in the pattern.
3. **Substitute a and b in the appropriate pattern.**
   - Don't forget to check the factors by multiplying!

03.04 Lesson Summary

A quadratic equation is an equation of degree 2. The graph of a quadratic equation is in the shape of a parabola which looks like an arc. The general form of the equation is represented by

$$f(x) = a(x – h)^2 + k$$

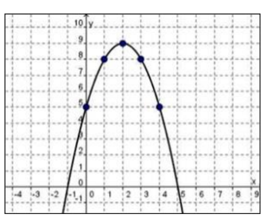
The vertex, or turning point, of the parabola is an ordered pair represented by (h, k). The vertex of the quadratic equation $f(x) = –(x – 2)^2 + 9$ is (2, 9).
Using the average rate of change, it is possible to find the slope between two points and observe if the graph's rate of change is increasing or decreasing.

The axis of symmetry is a line which splits the parabola in half so that both halves are symmetrical to one another. The equation for this line is found by setting the expression inside the parentheses equal to 0 and solving for x.

$$f(x) = –(x – 2)^2 + 9$$
$$x – 2 = 0$$
$$+2 \quad \underline{\ } +2$$
$$x = 2$$

The domain of the quadratic equation is the set of numbers which can be substituted for x and result in a unique value for y. In the case of the equation shown here, the domain is "all real numbers".

The range of the quadratic equation is the set of numbers that are produced from the domain values of x. Algebraically, determine whether the vertex is the minimum or maximum point of the graph. If a is positive, the vertex is the minimum point, and the range is all of the y-coordinates greater than or equal to the y-coordinate of the vertex. If a is negative, the vertex is the maximum point, and the range is all of the y-coordinates less than or equal to the y-coordinate of the vertex.



Graphically, it is easy to see the vertex is a maximum point and y-coordinates greater than 9 do not have a corresponding x-coordinate. Therefore, in the case of the equation shown, the range is $y \leq 9$.

The intercepts of a quadratic equation are the places on the graph where the parabola crosses the x- or y-axis. The x-intercepts are found by looking at the graph of the parabola where it crosses the x-axis.

The standard form of the quadratic equation is represented by

$$f(x) = ax^2 + bx + c$$

While the domain, range and intercepts of the parabola may be found in the same way as the general form of the parabola, the axis of symmetry and vertex must be found using the formula

$$x = -\frac{b}{2a}$$

The axis of symmetry is found first using this equation.

$$f(x) = x^2 + 6x + 8$$

$$x = -\frac{6}{2(1)}$$

$$x = -\frac{6}{2}$$

$$x = -3$$

Then the vertex is found by substituting the x-coordinate of –3 in the original equation and solving for y to find the y-coordinate.

$$f(x) = x^2 + 6x + 8$$
$$f(x) = (-3)^2 + 6(-3) + 8$$
$$f(x) = 9 + 6(-3) + 8$$
$$f(x) = 9 - 18 + 8$$

$$f(x) = -1 \quad (-3, -1)$$

03.06 Lesson Summary

Steps for Changing a Quadratic Equation from Standard Form to General Form

1. Isolate the terms containing the x variable.

2. Complete the square and balance the equation.
   - Factor the numerical GCF to ensure the leading coefficient of the binomial is 1.
   - Divide the coefficient of x by 2 and square the result to create a perfect square trinomial.
   - To keep the equation balanced, the constant added to one side of the equation must also be added to the other side of the equation.

3. Simplify.
   - Combine like terms.

- Factor the perfect square trinomial.
- Isolate the y variable.

4. Check your work.
   - Expand and multiply the squared binomial and then combine like terms.

Check the Discriminant to find the nature of the solutions

The Discriminant is $b^2 - 4ac$.

Steps for Solving a Quadratic Equation by Completing the Square

1. Isolate the terms containing the x variable.

2. Complete the square and balance the equation.
   - Divide the coefficient on x by 2 and square the result to create a perfect square trinomial.
   - To keep the equation balanced, the constant added to one side of the equation must also be added to the other side of the equation.
3. Simplify.
   - Combine like terms.
   - Factor the perfect square trinomial.
4. Solve for the variable.

03.07 Lesson Summary

Steps for Solving a Quadratic Equation by Factoring

1. Write the equation in standard form.
2. Factor the quadratic expression.
3. Set each factor equal to 0 and solve.
4. Check all solutions.

Steps for Solving a Quadratic Equation using the Quadratic Formula

$33x^2 - 12x = 27$

1. Rewrite the equation in standard form.

   $3x^2 - 12x = 27$
   $\underline{-27 \qquad -27}$
   $3x^2 - 12x - 27 = 0$
2. Factor any existing GCF.

   $3(x^2 - 4x - 9) = 0$
3. Apply and simplify the quadratic formula.

   $3(x^2 - 4x - 9) = 0$

   $a = 1$, $b = -4$ and $c = -9$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-9)}}{2(1)}$$

$$x = \frac{-(-4) \pm \sqrt{16 - 4(1)(-9)}}{2(1)}$$

$$x = \frac{4 \pm \sqrt{16 + 36}}{2}$$

$$x = \frac{4 \pm \sqrt{52}}{2}$$

$$x = \frac{4 \pm \sqrt{4} \bullet \sqrt{13}}{2}$$

$$x = \frac{4 \pm 2\sqrt{13}}{2}$$

$$x = \frac{4}{2} \pm \frac{2\sqrt{13}}{2}$$

$$x = 2 \pm \sqrt{13}$$

03.08 Lesson Summary

Steps for Solving a Quadratic Equation with Complex Numbers

Completing the Square

1. Isolate the terms containing the x variable.
2. Complete the square and balance the equation.
   - Factor the numerical GCF to ensure the leading coefficient on x^2 is 1.
   - Divide the coefficient on x by 2 and square the result to create a perfect square trinomial.
   - To keep the equation balanced, the constant added to one side of the equation must also be added to the other side of the equation.
3. Simplify.
   - Factor the perfect square trinomial.
   - Combine like terms.
4. Solve for the variable.

Quadratic Formula

1. Write the equation in standard form.
2. Factor the GCF.
3. Apply and simplify the quadratic formula.

03.09 Lesson Summary

Steps for Deriving a Quadratic Equation from a Parabola Using Distances.

Step 1
Substitute information into combined distance formulas. The right side of the equation is what was derived for the directrix.
$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(y_2 - y_1)^2}$

Step 2
Remove radicals.

Step 3
Distribute y term binomials.

Step 4
Simplify and isolate x terms.

Step 5
Isolate the y term.

Steps for Deriving a Quadratic Equation from a Parabola Using Focal Length.

Step 1
Sketch a graph to locate vertex and focal lengths.

Step 2
Solve for the a coefficient.
$\frac{1}{4p} = a$

Step 3
Substitute vertex and a coefficient into vertex form.

04.01 Lesson Summary

Steps to Dividing Polynomials

The steps used in this lesson to divide polynomials were:

1. Divide
2. Multiply
3. Subtract
4. Bring down
5. Repeat

Missing terms

When either polynomial does not have all terms according to the descending powers on the variable, it is necessary to add in terms with 0 coefficients.

For example:

- $x^3 + 3x + 2$ would become $x^3 + 0x^2 + 3x + 2$
- $x^4 - 1$ would become $x^4 + 0x^3 + 0x^2 + 0x - 1$

Remainders

- If the quotient has no remainder (a remainder of 0), then the divisor is said to be a factor of the dividend.

04.02 Lesson Summary

Steps to Polynomial Synthetic Division

Let $f(x) = x^3 + 3x^2 + 25$ and $g(x) = x + 5$. Find $\frac{f(x)}{g(x)}$.

$$\frac{f(x)}{g(x)} = \frac{x^3 + 3x^2 + 25}{x + 5}$$

Another way of representing this situation is $(x^3 + 3x^2 + 25) \div (x + 5)$

1. Set up the synthetic division problem.
Place the opposite constant of the divisor, −5, in the top left corner of the synthetic division problem.

$$\underline{-5\,|}$$

Write the coefficients of the dividend in descending order along the top row, making sure to insert 0x between $3x^2$ and 25 to hold the place value.

$$1x^3 + 3x^2 + 0x + 25$$

$$\underline{-5\,|}\quad 1 \quad 3 \quad 0 \quad 25$$

Underneath those coefficients, skip a line and then draw a line.

$$\begin{array}{r|rrrr} -5 & 1 & 3 & 0 & 25 \\ & & & & \\ \hline \end{array}$$

2. Follow the synthetic division process by multiplying and adding.

$$\begin{array}{r|rrrr} -5 & 1 & 3 & 0 & 25 \\ & \downarrow & -5 & 10 & -50 \\ \hline & 1 & -2 & 10 & -25 \end{array}$$

3. Identify the quotient.

$$\frac{f(x)}{g(x)} = \frac{x^3 + 3x^2 + 25}{x + 5} = x^2 - 2x - 10 - \frac{25}{x + 5}$$

Remainders

Remember that a remainder of 0 in the quotient means that the divisor is a factor of the dividend.

04.03 Lesson Summary

Understanding Theorems

A theorem is a proven statement based on experimentation. The three theorems in this lesson (The Fundamental Theorem of Algebra, The Factor Theorem, and The Remainder Theorem) will help you solve polynomial equations.

The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra generally states that the degree of a polynomial is equivalent to the number of zeros (both real and complex) of a function.

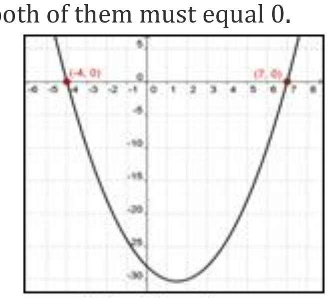
By the Fundamental Theorem of Algebra, the polynomial function $f(x)=x^2-3x-28$ has two zeros since the degree of the function is two. To determine these zeros, replace the function notation of $f(x)$ with zero and solve by factoring.

$$\begin{aligned} f(x) &= x^2-3x-28 \\ 0 &= x^2-3x-28 \\ 0 &= (x+4)(x-7) \end{aligned}$$

The zero product property tells us that for these factors to result in a product of 0, one or both of them must equal 0.

$x + 4 = 0$ and $x - 7 = 0$

The zeros of the function are $x=-4$ and $x=7$. The x-intercepts of the graph of this function confirm the zeros.



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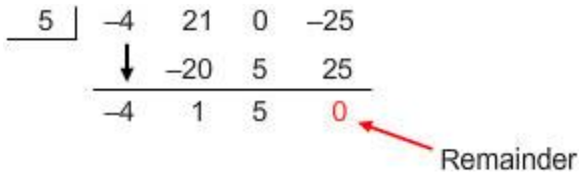
The Factor Theorem

The Factor Theorem states that a first degree binomial is a factor of a polynomial function if the remainder, when the polynomial is divided by the binomial, is zero.

To determine whether x–5 is a factor of the function $f(x)=-4x^3+21x2-25$, set up a synthetic division problem whereby$-4x^3+21x^2-25$ is divided by x–5.

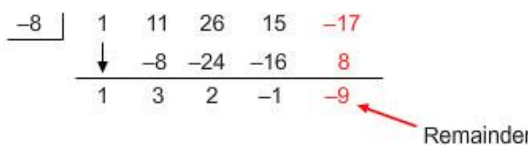


When the function $f(x)=-4x^3+21x^2-25$ is divided by the binomial x–5, the remainder is zero. So, x–5 is a factor of the function.

The Remainder Theorem

The Remainder Theorem states that when the opposite of the constant from the binomial divisor is substituted into a function for x, the result is the remainder.

When the polynomial function $f(x)=x^4+11x^3+26x^2+15x-17$ is divided by x+8 using synthetic division, the remainder is the last integer on the bottom row.



When the opposite constant in the divisor is substituted into the function, the result will be the same as the remainder in the synthetic division process.

$$f(x) = x^4+11x^3+26x^2+15x-17$$

$$f(-8) = (-8)^4+11(-8)^3+26(-8)^2+15(-8)-17$$

$$f(-8) = 4096+11(-512)+26(64)+15(-8)-17$$

$$f(-8) = 4096-5632+1664-120-17$$

$$f(-8) = -9$$

What it all means

Given a polynomial function f(x) and a number a, if (x–a) is a factor of f(x), then a is a zero of the polynomial.

The binomial (x–a) can be proved as a factor of f(x) by:

- Using Long division with (x–a) as the divisor
- Using Synthetic dvision with a as the divisior
- Using Factoring methods when appropriate (grouping, completing the square,…)

If a is a zero of the polynomial function f(x), then:

- The graph of f(x) crosses the x axis at (a,0).
- Substituting a into (x–a) will equal 0.
- Substituting a into f(x) will result in f(a)=0.

4.04 Lesson Summary

The Rational Root Theorem

The application of this theorem provides a list of all the possible rational zeros of a function.

$$f(x)=3x^3+x^2+21x-11$$

Step 1: Find the factors of p.

- The constant term −11 is p. Therefore, the factors of p are ±1 and ±11.

Step 2: Find the factors of q.

- The leading coefficient 3 is q. Three is divisible by ±1 and ±3.

Step 3: Write all factors of $\frac{p}{q}$.

- Write each of the factors of p and q as fractions with the factors of p as the numerator and the factors of q as the denominator.
- Factors of p: ±1 and ±11
- Factors of q: ±1 and ±3
- Factors of p/q

: ±1, ±11, ±1/3, ±11/3

Descartes' Rule of Signs

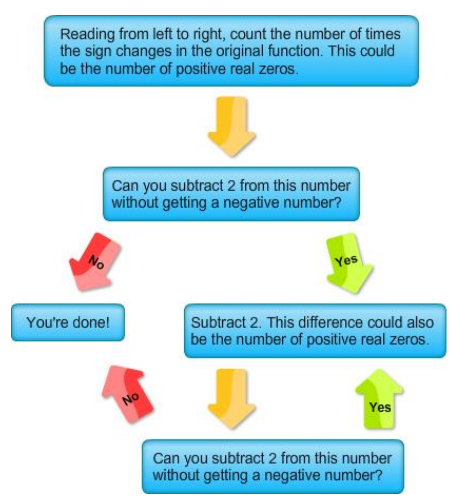
Descartes' Rule of Signs identifies the number of positive real, negative real and complex zeros of a given function.

$$f(x)=-7x^5-10x^4+3x^3+4x^2-13x+9$$

Positive Real Zeros

$f(x) = -7x^5 - 10x^4 + 3x^3 + 4x^2 - 13x + 9$

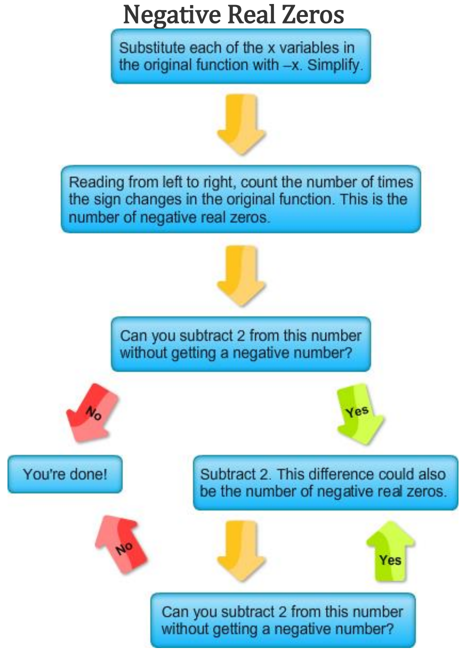
– – + + – +



Number of Sign Changes	Descartes Rule: Subtract 2	Result
3	3-2	1

There are 3 or 1 positive real zeros.

Negative Real Zeros



$$f(x) = -7x^5-10x^4+3x^3+4x^2-13x+9$$

$$f(-x) = -7(-x)^5-10(-x)^4+3(-x)^3+4(-x)^2-13(-x)+9$$

$$f(-x) = 7x^5-10x^4-3x^3+4x^2+13x+9$$

$$f(-x) = 7x^5-10x^4-3x^3+4x^2+13x+9$$

$$+ \; - \; - \; + \; + \; +$$

Number of Sign Changes	Descartes Rule: Subtract 2	Result
2	2-2	0

There are 2 or 0 negative real zeros.

To find the number of complex zeros, take the total number of zeros as indicated by the degree of the function and subtract all possible numbers of positive and negative real zeros.

Total	**5**	**5**	**5**	**5**
Positive	3	1	3	1
Negative	2	2	0	0
Complex	0	2	2	4

The function f(x)=–7x5–10x4+3x3+4x2–13x+9 has 3 or 1 positive real zeros, 2 or 0 negative real zeros, and 4 or 2 or 0 complex zeros.

04.05 Lesson Summary

Solving Polynomial Equations

$$f(x) = x^4 - 9x^3 + 15x^2 - 9x + 14$$

1. Determine the possible rational zeros

$$\text{Factors of } \frac{p}{q}: \pm 1, \pm 2, \pm 7, \pm 14$$

2. Determine the possible number of positive, negative, and complex zeros.

$$f(x) = x^4 - 9x^3 + 15x^2 - 9x + 14$$

There are 4, 2, or 0 positive zeros.

$$f(-x) = x^4 + 9x^3 + 15x^2 + 9x + 14$$

There are 0 negative zeros.

There are 4, 2, or 0 complex zeros.

3. Determine the zeros of the function.

   Use the Factor Theorem and synthetic division to test the possible solutions. Begin with the positive possible rational solutions.

$$\begin{array}{r|rrrrr} 1 & 1 & -9 & 15 & -9 & 14 \\ & \downarrow & 1 & -8 & 7 & -2 \\ \hline & 1 & -8 & 7 & -2 & 12 \end{array}$$

2	1	–9	15	–9	14
	↓	2	–14	2	–14
	1	–7	1	–7	0

Use the depressed equation of $0 = x^3 - 7x^2 + x - 7$ to continue testing the factors of $\frac{p}{q}$.

7	1	–7	1	–7
	↓	7	0	7
	1	0	1	0

Depressed equation: $0 = x^2 + 0x + 1$

Now that the depressed equation is a quadratic equation, it can be solved without testing any more possible rational solutions.

$x2+1=0$

$-1 \quad -1$

$x2=-1$

$\sqrt{x^2} = \sqrt{-1}$

$x = \sqrt{-1}$

$x = \pm i$

The solutions of the equation (or zeros of the function) are $x = 2$, $x = 7$, $x = i$, and $x = -i$.

Writing Polynomial Equations

Find the polynomial function with roots 6, –8, and –3.

1. Determine the factors of the function

   $(x - 6)(x + 8)(x + 3) = 0$

2. Multiply the factors

   $(x-6)(x+8)(x+3)=0$

   $X^3 +5x^2-42x-144=0$

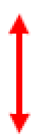
   $x^3+5x^2-42x-144=f(x)$

04.07 Lesson Summary

A polynomial function is a function composed of one or more terms, at least one of which contains a variable. The graph will pass the vertical line test. A polynomial function whose degree is three is called a cubic function. A polynomial function whose degree is four is called a quartic function.

Simple polynomial functions may be graphed by hand by substituting x-coordinates and solving for the corresponding y-coordinates.

Most polynomial functions are far more complicated and should be graphed using graphing technology such as GeoGebra or other graphing software. To graph a polynomial function using GeoGebra, enter the function into the Input box at the bottom of the screen.

<table>
<tr><th colspan="4">End Behavior of a Graph of a Polynomial Function</th></tr>
<tr><td colspan="2">Odd Degree</td><td colspan="2">Even Degree</td></tr>
<tr><td colspan="2">The ends will travel in opposite directions.</td><td colspan="2">The ends will travel in the same direction.</td></tr>
<tr><td>Positive Leading Coefficent</td><td>Negative Leading Coefficient</td><td>Positive Leading Coefficient</td><td>Negative Leading Coefficient</td></tr>
<tr><td>Left side continues down, right side continues up
</td><td>Left side continues up, right side continues down
</td><td>Both sides continue up.
</td><td>Both sides continue down.
</td></tr>
</table>

The real number solutions of a polynomial equation can be found by identifying the x-intercepts from the graph of the function. Graph the function using graphing technology and find the point(s) of intersection between the graph and the x-axis.

Average Rate of Change

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Increasing versus Decreasing

- If the average rate of change is positive, then the function is considered to be increasing.
- If the average rate of change is negative, then the function is considered to be decreasing.

Minimums and Maximums

- Minimums are located on the lowest point of a graph. A local minimum is the bottom of a “valley” or turning point where the graph goes from decreasing to increasing.

- Maximums are located on the highest point of a graph. A local maximum is the top of a “hill” or turning point where the graph goes from increasing to decreasing.

Translations

- f(x – h) is a horizontal shift of h units.

- f(x) + k is a vertical shift of k units.

- f(ax) affects the width of the graph. A negative a will reflect the graph over the y–axis.

- a • f(x) affects the width of the graph and possible vertical and horizontal shifts. A negative a will reflect the graph over the x–axis.

04.08 Lesson Summary

Algebraic Proofs

- Polynomial identities can be proven to be true by simplifying the identity through application of algebraic theorems and principles.
- Start with the side of the identity that can be simplified the easiest.
- Sometimes, following a “clue” will lead to a dead–end in your proof. Do not give up. Just follow a different “clue.” The more practice you have with proofs, the more you will be able to predict the dead ends.

Application to Numerical Relationships

Polynomial identities apply to more than just polynomials. Replacing the variables with numbers can help prove numerical relationships as well.

5.01 Lesson Summary

Simplifying Rational Expressions

$\underline{x^2+6x-27}$

/x+9

1. **Factor the numerator and the denominator, if possible**
   $\underline{(x-3)(x+9)}$

   x+9

2. **Identify the restrictions.**
   x+9 = 0

   $\underline{-9}$ __ $\underline{-9}$

   x = −9

3. **Simplify.** $\dfrac{(x-3)\cancel{(x+9)}^{1}}{\cancel{x+9}_{1}}$
   x−3

4. x – 3 and x ≠ −9

5.02 Lesson Summary

Multiplying Rational Expressions

1. Factor all expressions.
2. Simplify what can be simplified.
3. Multiply the simplified factors together.

Dividing Rational Expressions

1. Convert the division expression to multiplication by the reciprocal.
2. Keep the first rational expression, change the division to multiplication, then flip the second rational expression.
3. Factor all expressions.
4. Simplify what can be simplified.
5. Multiply the simplified factors together.

5.03 Lesson Summary

Adding and Subtracting Rational Expressions with Like Denominators

When rational expressions have like denominators, combine the like terms in the numerators. Then, simplify, if possible.

Adding and Subtracting Rational Expressions with Unlike Denominators

1. Factor the denominators.
2. Find a common denominator.
3. Combine the rational expressions.
4. Simplify.

5.04 Lesson Summary

Simplifying Complex Fractions

Method 1:

1. Rewrite the complex fraction as a division expression.
2. Simplify the expressions inside parentheses.
3. Divide.

Method 2:

1. Find the least common denominator.
2. Find common factors and simplify.

Remember, dividing by a fraction is the same as multiplying by the reciprocal of the divisor.

5.06 Lesson Summary

A discontinuity of a rational function is a point on the graph of a function where the function does not exist. This point of discontinuity may be found algebraically, by first simplifying the function, and then finding the restrictions on the cancelled common factor(s).

A point of discontinuity is indicated on a graph by an open circle.

The zeros of a rational function may be found by substituting 0 for f(x) and solving for x. This is most easily done using the simplified rational function.

5.07 Lesson Summary

Asymptotes of Rational Functions
Asymptotes are imaginary lines on a graph that a function or equation approaches more and more closely.

Vertical Asymptotes
The location of a vertical asymptote is found by setting the denominator of a simplified function equal to 0 and solving for x.

Horizontal Asymptotes
A horizontal asymptote is found by comparing the degree of the numerator to the degree of the denominator.

Relationship between the Degrees of the Numerator and Denominator of the Rational Function f(x)	Horizontal Asymptote
The degree of the numerator is smaller than the degree of the denominator.	$y = 0$
The degree of the numerator and denominator are the same.	$y = \frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}}$
The degree of the numerator is greater than the degree of the denominator.	No Horizontal Asymptote

Oblique Asymptotes
An oblique asymptote, which is a slanted asymptote, is found by dividing the numerator of the rational function by its denominator. Any remainder in the quotient may be disregarded.

By definition the oblique asymptote is found when the degree of the numerator is one more than the degree of the denominator, and there is no horizontal asymptote when this occurs. Therefore, the graph of a function cannot have both a horizontal asymptote and an oblique asymptote.

5.08 Lesson Summary

Solving Rational Equations

1. Find a common denominator.
2. Simplify the equation.
3. Solve for the variable.
4. Check your work.

5.09 Lesson Summary

Work Problems

Work problems are problems that apply rational equations to the real world.